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# Controlled Diffusions with Constraints, II

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We consider the ergodic control for a controlled nondegenerate diffusion when $m$ other ($m$ finite) ergodic costs are required to satisfy prescribed bounds. Under a condition on the cost functions that penalizes instability, the existence of an optimal stable Markov control is established by convex analytic arguments. © 1993 Academic Press, Inc.

## I. INTRODUCTION

In [5], the existence of an optimal stable Markov control was established for the ergodic control problem for a nondegenerate diffusion when $m$ other ($m$ finite) ergodic costs are required to satisfy prescribed bounds. This was done under a uniform stability assumption that required in particular that all Markov controls be stable (that is, the corresponding diffusions positive recurrent) and the set of corresponding invariant probability measures compact in Prohorov topology. This condition can be restrictive at times. In this note, we recover the same result under an alternative set of conditions. These allow for unstable Markov controls but put a condition on the cost functions which penalizes instability. The latter condition is often satisfied in practice, e.g., by the quadratic cost.

A precise statement of the problem is given in the next section. The main result is proved in Section IV following some preliminaries in Section III. We rely heavily on [5] for much detail. Reproducing the same here would be a lengthy affair and would add nothing new.

## II. NOTATION AND PROBLEM DESCRIPTION

Let $X(\cdot)=[X_1(\cdot), X_2(\cdot), \ldots, X_d(\cdot)]^T$ be a controlled diffusion satisfying

$$X(t)=X_0+\int_0^t m(X(s), u(s))\, ds+\int_0^t \sigma(X(s))\, dW(s), \qquad t\geqslant 0, \tag{2.1}$$



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where

(i) $m(\cdot,\cdot) = [m_1(\cdot,\cdot), ..., m_d(\cdot,\cdot)]^T: R^d \times U \to R^d$ ($U$ being a prescribed compact metric space) is bounded continuous and Lipschitz in its first argument uniformly with respect to the second;

(ii) $\sigma(\cdot) = [[\sigma_{ij}(\cdot)]] : R^d \to R^{d \times d}$ is bounded Lipschitz and satisfies that for some $\lambda > 0$,

$$x^T \sigma \sigma^T x \geqslant \lambda \, \|x\|^2, \qquad x \in R^d;$$

(iii) $X_0$ is a random variable with a prescribed law, $\pi_0$;

(iv) $W(\cdot) = [W_1(\cdot), ..., W_d(\cdot)]^T$ is a standard Wiener process independent of $X_0$;

(v) $u(\cdot)$ is a $U$-valued "control process" with measurable paths satisfying that for $t \geqslant s \geqslant y$, $W(t) - W(s)$ is independent of $u(z)$, $z \leqslant y$.

If $u(\cdot) = v(X(\cdot))$ for a measurable $v: R^d \to U$, we call it a Markov control. Under this control, (2.1) has an a.s. unique strong solution [13]. By abuse of terminology, the map $v$ itself may be referred to as the Markov control.

We assume the relaxed control framework of [7]. Thus $U = P(V)$ for a compact metric space $V$. (For a Polish space $X$, $P(X)$ denotes the Polish space of probability measures on $X$ with the Prohorov topology [2].) Also, $m(\cdot, \cdot)$ will be assumed to be of the form

$$m_i(x, u) = \int \bar{m}_i(x, y) \, u(dy), \qquad x \in R^d, u \in U,$$

for some $\bar{m}(\cdot, \cdot) = [\bar{m}_1(\cdot, \cdot), ..., \bar{m}_d(\cdot, \cdot)]^T : R^d \times V \to R^d$ which is bounded continuous and Lipschitz in its first argument uniformly with respect to the second. If $u(\cdot)$ (or $v(\cdot)$) take values in the set of Dirac measures on $V$, we refer to them as precise (resp. precise Markov) controls.

A Markov control $v$ will be said to be stable if the corresponding $X(\cdot)$ is positive recurrent and hence has a unique invariant probability measure $\eta[v]$ [1]. Define $\pi[v] \in P(R^d \times V)$ by

$$\pi[v](dx, du) = \eta[v](dx) \, v(x)(du).$$

Let $G = \{\pi[v] \mid v \text{ a stable Markov control}\}$.

Let $k_i: R^d \times V \to R^+$, $0 \leqslant i \leqslant m$, be continuous maps and $\alpha_i > 0$, $1 \leqslant i \leqslant m$, prescribed scalars. Let $H \subset G$ be the set on which

$$\int k_i \, d\pi[v] \leqslant \alpha_i, \qquad 1 \leqslant i \leqslant m. \tag{2.2}$$

We assume that $H$ is nonempty and we require $\{k_i\}$ to satisfy

$$\liminf_{\|x\| \to \infty} \inf_u k_i(x, u) > \alpha_i, \qquad 0 \leqslant i \leqslant m, \tag{2.3}$$

where,

$$\alpha_0 = \inf_H \int k_0 \, d\pi[v]$$

is assumed to be finite. Inequality (2.3) is satisfied in particular for $\{k_i\}$ of the form $k_i(x, u) = f_i(x)$ where

$$f_i(x) \to \infty \qquad \text{as} \quad \|x\| \to \infty.$$

Our problem is to minimize over $H$ the cost

$$\int k_0 \, d\pi[v]. \tag{2.4}$$

Note the important differences between this and the problem studied in [5]. In [5], (2.3) was not assumed and the constraints studied were of a more general form

$$\beta_i \leqslant \int k_i \, d\pi[v] \leqslant \alpha_i, \qquad 1 \leqslant i \leqslant m$$

for suitable $\{\beta_i\}$, $\{\alpha_i\}$. Instead, a Liapunov-type stability condition was assumed [5, Assumption A, p. 90]. This condition ensured that the set of attainable $\pi[v]$ was compact and so was the set of feasible $\pi[u]$ obeying the constraints. In this paper we allow for the possibility that some Markov controls may in fact be unstable (i.e., lead to null recurrence or transience). However, (2.3) penalizes unstable behaviour by putting large costs on large values of state. As we prove below, this automatically ensures existence of optimal stable Markov controls. The present conditions are more useful than those of [5] because the strong uniform stability implicit in Assumption A of [5] may be hard to obtain, but costs penalizing large values of the state are not uncommon.

## III. PRELIMINARIES

This section establishes some technical lemmas needed later.

LEMMA 3.1. *Let $v_1$, $v_2$ be Markov controls such that $v_1(x) = v_2(x)$ when $\|x\| > r$ for some $r > 0$. If $v_1$ is stable, so is $v_2$.*

*Proof.* Let $\infty > r_2 > r_1 > r$, $B_i = \{x \in R^d \mid \|x\| \leqslant r_i\}$, $\partial B_i$ the boundary of $B_i$, and $\tau_i = \inf\{t \geqslant 0 \mid X(t) \in \partial B_i\}$ for $i = 1, 2$. Consider $X(\cdot)$ controlled by $v_2$. Since the function

$$f(x) = E[\tau_1 / X(0) = x]$$

for $\|x\| \geqslant r_1$ is the same under either $v_1$ or $v_2$ and $v_1$ is stable, it is finite under $v_2$ [1]. Now $f$ satisfies the p.d.e.

$$\sum_{i=1}^{d} m_i(x, v_2(x)) \frac{\partial f}{\partial x_i} + (1/2) \sum_{i,j,k} \sigma_{ij}(x)\, \sigma_{jk}(x) \frac{\partial^2 f}{\partial x_i\, \partial x_j} = -1 \tag{3.1}$$

for $\|x\| > r_1$, $x = [x_1, ..., x_d]$. From standard regularity theory for elliptic p.d.e. [9, Chap. 5], it follows that $f$ is continuous and thus its maximum on $\partial B_2$ is obtained. Thus

$$\sup_{x \in \partial B_2} E[\tau_1 / X(0) = x] < \infty. \tag{3.2}$$

From Lemma 3.2 in [4, p. 36], we have

$$\sup_{x \in \partial B_1} E[\tau_2 / X(0) = x] < \infty. \tag{3.3}$$

Therefore, for

$$\bar{\tau} = \inf\{t > 0 \mid X(t) \in \partial B_1,\ X(s) \in \partial B_2 \text{ for some } s \in [0, t]\},$$

we have

$$\sup_{X \in \partial B_1} E[\bar{\tau} / X(0) = x] < 0.$$

It is proved in [8] that this implies positive recurrence. Q.E.D.

LEMMA 3.2. *The set* $D = \{\eta[v'] \mid v'(x) = v(x)$ *for* $\|x\| > r\}$ *is tight*, $v$ *being as in Lemma* 3.1.

*Proof.* Let $N > r_2 > r_1 > r$, $\{\tau_i\}$, $\{B_i\}$, $\{\partial B_i\}$ be as above, and $\tau = \inf\{t \geqslant \tau_2 \mid X(t) \in \partial B_1\}$. Then for $\eta[v'] \in D$,

$$\int I\{|x| > N\}\, d\eta[v'](x) = E\left[\int_0^{\tau} I\{|X(t)| > N\}\, dt\right] \Big/ E[\tau]$$

for $X(\cdot)$ controlled by $v'$, the initial law being a certain probability measure supported on $\partial B_1$. (See, e.g., [4, Lemma 2.2, p. 150].) Thus

$$\int I\{|x|>N\}\, d\eta[v'](x)$$

$$\leqslant \frac{\sup_{x\in\partial B_2} E[\int_0^{\tau_1} I\{X(t)>N\}\, dt/X(0)=x]}{\inf_{x\in\partial B_2} E[\tau_1/X(0)=x]}, \tag{3.4}$$

where we use the fact that $I\{\|x\|>N\}$ vanishes on $B_2$. An argument analogous to the one leading to (3.2) above shows that the infimum in the denominator is attained and hence strictly positive. Also, both the numerator and the denominator are unchanged if we replace $v'$ by $v$. Let $N\to\infty$. The quantity

$$\phi_N(x)=E\left[\int_0^{\tau_1} I\{X(t)\geqslant N\}\, dt/X(0)=x\right]$$

(with $X(\cdot)$ now controlled by $v$) decreases monotonically to zero as $N\to\infty$. Also, $\phi_N$ satisfies a p.d.e. similar to (3.1) for $\|x\|>r_1$ with the right hand side of (3.1) changed to $-I\{\|x\|\geqslant N\}$. Again, standard regularity theory for elliptic p.d.e. tells us that $\phi_N$ are continuous on $\{\|x\|>r_1\}$. By Dini's theorem, $\phi_N(x)$ decreases to zero uniformly for $x\in\partial B_2$ as $N\to\infty$. Thus the right hand side of (3.4) goes to zero as $N\to\infty$ uniformly in $\eta[v']\in D$. The claim follows. Q.E.D.

For twice continuously differentiable $f\in C(R^d)$, let

$$(Lf)(x,y)=\sum_i \bar{m}_i(x,y)\frac{\partial f}{\partial x_i}$$

$$+(1/2)\sum_{i,j,k}\sigma_{ik}(x)\,\sigma_{jk}(x)\frac{\partial^2 f}{\partial x_i\,\partial x_j}.$$

LEMMA 3.3. *Let $\pi\in P(R^d\times V)$ disintegrate [12] as*

$$\pi(dx,dy)=\eta(dx)\,v'(x,dy)$$

*for $\eta\in P(R^d)$, $x\to v'(x,\cdot)$: $R^d\to U$, $v'$ being prescribed $\eta$ – a.s. Suppose*

$$\int Lf\, d\pi=0 \tag{3.5}$$

*for all smooth compactly supported $f$: $R^d\to R$. Then $\pi=\pi[v]$ for $v$ defined by $v(x)=v'(x,\cdot)$ for an arbitrary representative of the $\eta$-a.s. equivalence class $v'$ defined above.*

*Proof.* Disintegrate $\pi$ as

$$\pi(dx, dy) = \eta(dx)\, v'(x, dy),$$

where $\eta$ is the image of $\pi$ under the projection $R^d \times V \to R^d$ and $v' \colon R^d \to U$ is the regular conditional law defined $\eta$-a.s. Clearly $v(x) = v'(x, \cdot)$ can be identified with a Markov control. Let

$$(L_v f)(x) = \sum_i m_i(x, v(x)) \frac{\partial f}{\partial x_i} + (1/2) \sum_{ijk} \sigma_{ik}(x)\, \sigma_{jk}(x) \frac{\partial^2 f}{\partial x_i\, \partial x_j}$$

for twice continuously differentiable $f \in C(R^d)$. Then (3.5) becomes

$$\int L_v f(x)\, \eta(dx) = 0. \tag{3.5'}$$

Thus

$$L_v^* \eta = 0$$

in the sense of distributions, $L_v^*$ being the formal adjoint of the operator $L_v$. It is proved in [4, Lemma 1.1, p. 144] that this along with the fact that $\eta \in P(R^d)$ implies that $\eta = \eta[v]$. Thus $\pi = \pi[v]$. Q.E.D.

LEMMA 3.4. *$G$ is closed convex and its extreme points correspond to precise Markov controls.*

*Proof.* As in Lemma 3.3, one observes that elements of $G$ are characterized by (3.5). It is easy to see from this that $G$ will be closed convex (See [5, Lemma 3.1] for details.) Let $v$, $v_1, v_2$ be Markov controls such that $v$ is stable, $v_1(x) \neq v_2(x)$ for $x$ belonging to a set $A$ of strictly positive Lebesgue measure, and $v(x) = av_1(x) + (1-a)\, v_2(x)$ for all $x$ and $a$ fixed $a \in (0, 1)$. The last claim will follow if we show that $\pi[v]$ cannot be an extreme point of $G$, because any relaxed Markov control $v$ which is not a precise Markov control can be written as a convex combination of two other relaxed Markov controls which differ on a set of strictly positive Lebesgue measure. (This is an easy consequence of the fact that a non-Dirac probability measure can be written as a convex combination of two distinct probability measures.) If $G$ is compact, this is proved in Lemma 3.3 of [5]. A sketch of the proof follows: One may assume without any loss of generality that the set of positive Lebesgue measure on which $v_1, v_2$ differ is bounded. For a relaxed Markov control $v_0$, define another relaxed Markov control $\bar{v}$ by

$$\begin{aligned} v(x) = {} & (b\phi[v_1](x)\, v_1(x) + (1-b)\, \phi[v_0](x)\, \bar{v}(x))/(b\phi[v_1](x) \\ & + (1-b)\, \phi[v_0](x)) \end{aligned}$$

for some $b \in (0, 1)$, $\phi[u]$ being the density of $\eta[u]$ with respect to the Lebesgue measure. (It is shown in [5] that this is possible for a suitable choice of $b$.) The map $\eta[v_0] \to \eta[\bar{v}]$ thus defined can be shown to have a fixed point, say $\eta[\tilde{v}]$, leading to

$$v(x) = (b\phi[v_1](x)\, v_1(x) + (1-b)\, \phi[\tilde{v}](x)\, \tilde{v}(x))/(b\phi[v_1](x) + (1-b)\, \phi[\tilde{v}]\,(x)).$$

It is easy to check now that

$$\pi[v] = b\pi[v_1] + (1-b)\, \pi[\tilde{v}],$$

proving the claim.

Suppose $G$ is not compact. Let $G(v) = \{\pi[v'] \mid v'(x) = v(x) \text{ for } \|x\| > R\}$. By Lemma 3.2 and Prohorov's theorem, $G(v)$ is relatively compact in $P(R^d \times V)$. Recall the topology on the space of Markov controls defined in [3]. It is clear that the set $\{v' \mid v'(x) = v(x) \text{ for } \|x\| > R\}$ is compact in this topology. The map $v \to \eta[v]$ is shown to be continuous in [5, pp. 95–96], with the topology of [3] on its domain and the total variation norm topology on its range. It is easy to check from this that the map $v \to \pi[v] \in P(R^d \times V)$ is continuous. Thus $G(v)$ is compact. It's convexity follows from a routine verification that convex combinations satisfy (3.5), along the lines of Lemma 3.1 of [5]. Argue as in Lemma 3.3 of [5] (a sketch of the argument is given above) to conclude that $\pi[v]$ cannot be an extreme point of $G(v)$ and hence of $G$. Thus extreme points of $G$ correspond to precise Markov controls. Q.E.D.

COROLLARY 3.1. *$H$ is closed convex.*

*Proof.* Convexity follows easily from the above lemma. Closedness follows upon observing that (2.2) is preserved under sequential limits in $G$. Q.E.D.

Given a probability measure $\mu$ on a Borel set $A$ of probability measures on a Polish space $S$, its barycenter $\bar{\mu}$ is the probability measure on $S$ given by

$$\int_A \left(\int_S f\, d\rho\right) \mu(d\rho) = \int_S f\, d\bar{\mu}, \qquad f \in C_b(S).$$

LEMMA 3.5. *Any element of $H$ is the barycenter of a probability measure supported on the extreme points of $H$.*

*Proof.* Let $\bar{R}^d = R^d \cup \{\infty\}$ denote the one point compactification of $R^d$. View $P(R^d \times V)$ as a subset of $P(\bar{R}^d \times V)$ by identifying each $\mu$ in the former with the unique $\bar{\mu}$ in the latter which restricts to $\mu$ on $R^d \times V$. Let

$\bar{H}$ be the closure of $H$ in $P(\bar{R}^d \times V)$. Then $\bar{H}$ is compact. Let $H_e$, $\bar{H}_e$ denote the sets of extreme points of $H$, $\bar{H}$, respectively. Suppose $v \in H_e \backslash \bar{H}_e$. Then $v$ is a convex combination of two distinct elements of $\bar{H}$ at least one of which must lie in $\bar{H} \backslash H$ and thus assign a strictly positive mass to $\{\infty\} \times V$. This is possible only if $v(\{\infty\} \times V) > 0$ which is false. Thus $H_e \subset \bar{H}_e$. By Choquet's theorem [11], each $\mu \in H$ is the barycenter of a probability measure $\varphi$ on $\bar{H}_e$. If $\varphi(\bar{H}_e \backslash H_e) > 0$, we must have $\mu(\{\infty\} \times V) > 0$, a contradiction. Thus $\varphi(H_e) = 1$ and the claim follows. Q.E.D.

LEMMA 3.6. *Each* $v \in \bar{H}$ *is of the form*

$$v(A) = \delta v'(A \cap (R^d \times V)) + (1 - \delta)\, v''(A \cap (\{\infty\} \times V)) \tag{3.6}$$

*for A Borel in* $\bar{R}^d \times V$, *where* $\delta \in [0, 1]$, $v' \in G$, *and* $v'' \in P(\{\infty\} \times V)$.

*Proof.* That (3.6) holds for some $v' \in P\{R^d \times V)$ is obvious. Without any loss of generality, let $\delta > 0$. If $\delta = 1$, $v \in H$ and the claim is immediate. If not, there exist $\{\pi[v_n]\}$ in $H$ such that $\pi[v_n] \to v$ in $P(\bar{R}^d \times V)$. Clearly,

$$\int Lf \, d\pi[v_n] = 0, \qquad n = 1, 2, ..., \tag{3.7}$$

for smooth compactly supported $f \in C(R^d)$, since $\eta[v_n]$ is the invariant probability measure under $v_n$, $n = 1, 2, ...$. Letting $n \to \infty$ in (3.7), we conclude that (3.7) holds for $v'$ in place of $\pi[v_n]$. (Recall that $\delta > 0$.) The claim now follows from Lemma 3.3. Q.E.D.

THEOREM 3.1. *Problem* (2.4) *attains its minimum at an extreme point of H.*

*Proof.* Let $\pi[v_n]$, $n \geqslant 1$, be a sequence in $H$ such that

$$\int k_0 \, d\pi[v_n] \downarrow \alpha_0.$$

Dropping to a subsequence if necessary, suppose that $\pi[v_n] \to v$ in $\bar{H}$ for a $v$ as in (3.6) with $v' = \pi[v]$ for some $v$. Fix $j$, $1 \leqslant j \leqslant m$. Pick $\varepsilon_j > 0$, $r_j > 0$ such that

$$\inf_u k_j(x, u) \geqslant \alpha_j + \varepsilon_j \qquad \text{for} \quad \|x\| \geqslant r_j.$$

For $n \geqslant 1$, pick $k_{jn} \in C_b(R^d \times V)$ such that $k_{jn} \leqslant k_j$ with

$$\begin{aligned} k_{jn}(x, u) &= k_j(x, u) && \text{for} \quad \|x\| \leqslant n,\ u \in V, \\ &= \alpha_j + \varepsilon_j && \text{for} \quad \|x\| \geqslant nVr_j + 1,\ u \in V. \end{aligned}$$

Then

$$\alpha_j \geqslant \liminf_{1 \to \infty} \int k_j \, d\pi[v_1]$$

$$\geqslant \lim_{1 \to \infty} \int k_{jn} \, d\pi[v_1]$$

$$= \delta \int k_{jn} \, d\pi[v] + (1-\delta)(\alpha_j + \varepsilon_j).$$

Let $n \to \infty$ on the right to obtain

$$\alpha_j \geqslant \delta \int k_j \, d\pi[v] + (1-\delta)(\alpha_j + \varepsilon_j).$$

This is possible only if $\delta > 0$ and

$$\int k_j \, d\pi[v] \leqslant \alpha_j.$$

Since $j$, $1 \leqslant j \leqslant m$, was arbitrary, $\pi[v] \in H$. Now let $\varepsilon > 0$ and $r > 0$ be such that

$$\inf_u k_0(x, u) > \alpha_0 + \varepsilon \qquad \text{for} \quad \|x\| \geqslant r.$$

Argue as above to conclude that

$$\alpha_0 \geqslant \delta \int k_0 \, d\pi[v] + (1-\delta)(\alpha_0 + \varepsilon)$$

and therefore,

$$\int k_0 \, d\pi[v] \leqslant \alpha_0.$$

From the definition of $\alpha_0$, equality holds. Thus (2.4) attains its minimum on $H$ at $\pi[v]$. By Lemma 3.4, $\pi[v]$ is the barycenter of a probability measure $\nu$ on $H_e$. Thus

$$\alpha_0 = \int_{H_e} d\nu(\rho) \left( \int k_0 \, d\rho \right).$$

Since $\int k_0 \, d\rho \geqslant \alpha_0$ for $\rho \in H_e$, we must have $\int k_0 \, d\rho = \alpha_0$ for $\nu$–a.s. $\rho$, proving the claim. Q.E.D.

## IV. Proof of the Main Result

Our main result is the following.

THEOREM 4.1. *The constrained problem above has an optimal precise Markov control.*

*Proof.* View $\bar{G}$, $\bar{H}$ as subsets of the topological vector space of finite signed measures on $\bar{R}^d \times V$. Let $\pi[v] \in H_e$ be such that $\int k_0 \, d\pi[v] = \alpha_0$. If it is an extreme point of $\bar{G}$, it is also an extreme point of $G$, otherwise it would assign strictly positive probability to $\{\infty\} \times V$. The claim then follows from Lemma 3.3. Suppose not. We then need the following lemma which is a special case of Dubins [6]. (See also [14, p. 265].) The proof is included because the ideas therein will also be needed later.

LEMMA 4.1. *$\pi[v]$ is a convex combination of some $k \leqslant m+1$ distinct extreme points of $\bar{G}$.*

*Proof.* Suppose not. Then it must be in the relative interior of an $n$-simplex $P$ for some $n \geqslant m+2$, formed by $n$ distinct extreme points of $\bar{G}$. Let $M$ be the $(n-1)$-dimensional hyperspace spanned by $P$ and $B$ an open ball in $M$ contained in the interior of $P$ and centered at $\pi[v]$. $P$ and hence $B$ are in $\bar{G}$. Consider the intersections of constraint hyperplanes $\{v \mid \int k_i \, dv = \alpha_i\}$, $1 \leqslant i \leqslant m$, with $M$. Since at most $m$ of them intersect $B$, the dimension in $M$ of their further intersections containing $\pi[v]$ is at least one and thus cannot have a corner in the interior of $B$. Thus $\pi[v]$ cannot be an extreme point of $\bar{H}$, a contradiction. The claim follows. Q.E.D.

If any of these extreme points assigned strictly positive probability to $\{\infty\} \times V$, so would $\pi[v]$. Thus they must be extreme points of $G$ itself. By Lemma 3.3, they are of the form $\pi[v_i]$, $1 \leqslant i \leqslant k$, for some precise Markov controls $\{v_i\}$. Let $A \subset R^d$ be a bounded set of strictly positive Lebesgue measure such that for a.e. $x \in A$, at least two of the $\{v_i(x)\}$ differ from each other. Write $A$ as the union of disjoint sets $A_i$, $1 \leqslant i \leqslant n$, of strictly positive Lebesgue measure. On $A$ (and hence on each $A_i$), $v(x)$ is a.e. a convex combination of at least two distinct elements of $\{v_i(x),\ 1 \leqslant i \leqslant k\}$ and thus cannot be a.s. Dirac. Argue as in Lemma 3.3 of [5] or Lemma 3.4 above to conclude that for each $i$,

$$\pi[v] = a_i \pi[v_{i1}] + (1 - a_i)\, \pi[v_{i2}]$$

for some $a_i \in (0, 1)$ and stable Markov controls $v_{i1}$, $v_{i2}$ which agree with

$v$ a.e. on $A_i^c$ and differ from each other and hence from $v$ a.e. on $A_i$. Furthermore,

$$v(x) = (a_i \phi[v_{i1}](x)\, v_{i1}(x) + (1-a_i)\, \phi[v_{i2}](x)\, v_{i2}(x))/(a\phi[v_{i1}](x) + (1-a)\, \phi[v_{i2}]\,(x)).$$

The line segment $L_i$, joining $\pi[v_{i1}]$, $\pi[v_{i2}]$ has $\pi[v]$ in its relative interior.

LEMMA 4.2. *Each of the $L_i$'s is transversal to the linear span of the rest.*

*Proof.* Suppose not. For simplicity, take the specific case of $L_1 \in \operatorname{span}(L_2, L_3)$. Then there exist $\pi[u_i] \in L_i$, $1 \leqslant i \leqslant 3$, $\pi[u_i] \neq \pi[v]$, such that $\pi[u_i] \neq \pi[v_{ij}]$ for $j = 1, 2$, and

$$\pi[u_1] = a\pi[u_2] + (1-a)\, \pi[u_3]$$

for some $a \in [0, 1]$. As in Lemma 3.1 of [5],

$$u_1(x) = (a\phi[u_2](x)\, u_2(x) + (1-a)\, \phi[u_3](x)\, u_3(x))/(a\phi[u_2](x) + (1-a)\, \phi[u_3](x)) \quad \text{a.e.} \tag{4.1}$$

By a similar argument,

$$u_1(x) = (b\phi[v_{11}](x)\, v_{11}(x) + (1-b)\, \phi[v_{12}](x)\, v_{12}(x))/(b\phi[v_{11}](x) + (1-b)\, \phi[v_{12}](x))$$

a.e. for some $b \in (0, 1)$. Since $v_{11}(x)$, $v_{12}(x)$ differ from each other and from $v(x)$ a.e. on $A_1$, so does $u_1(x)$. But $v_{ij}(x)$ for $i = 2, 3$ and $j = 1, 2$ agree with $v(x)$ a.e. on $A_1$ and therefore, by an argument analogous to the above, so do $u_2(x)$, $u_3(x)$. But (4.1) implies that $v(x)$, $u_1(x)$ agree a.e. on $A_1$, a contradiction. The claim follows. Q.E.D.

The polytope formed by end points of the $L_i$'s contains $\pi[v]$ in its relative interior and is in $G$. This leads to a contradiction similar to the one in the proof of Lemma 4.1 for $n$ sufficiently large, in view of Lemma 4.2. Thus $\pi[v]$ must be an extreme point of $G$ itself and we are done.

*Remarks.* We take this opportunity to insert a corrigendum for [5]. Reference [5] proves a similar result for $m = 1$ under the stability hypotheses mentioned earlier and suggests that the same argument can be iterated to get the result for $m > 1$. The latter is not quite true. The above proof, however, applies equally well to the set-up of [5] and may be used to prove the $m > 1$ case there. Another minor detail: The range of $a(x)$, $b(x)$ in Lemma 4.4 of [5] and the paragraph preceding it should be $(0, 1)$ and not $[0, 1]$ as printed there.

The following is now immediate from standard Lagrange multiplier theory [10, pp. 216–219].

THEOREM 4.2. *If the inequalities in* (2.2) *are strict at some point in* $H$, *there exist* $\lambda_1, \lambda_2, ..., \lambda_m \geqslant 0$ *such that the map*

$$v \to \int k_0 \, dv + \sum_{i=1}^{m} \lambda_i \left( \alpha_i - \int k_i \, dv \right) = F(v, \lambda_1, ..., \lambda_m)$$

*attains its minimum on* $G$ *at the* $\pi[v]$ *of Theorem* 4.1. *Moreover, for all* $v \in G$, $\mu_1, ..., \mu_m \geqslant 0$,

$$F(v, \lambda_1, ..., \lambda_m) \geqslant F(\pi[v], \lambda_1, ..., \lambda_m) \geqslant F(\pi[v], \mu_1, ..., \mu_m).$$

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